

Time-Optimal Control of Linear Fractional Systems

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Abstract

Problem of time-optimal control of linear systems with fractional dynamics is treated in the paper from the convex-analytic standpoint. A linear system of fractional differential equations involving Riemann–Liouville derivatives is considered. A method to construct a control function that brings trajectory of the system to the terminal state in the shortest time is proposed in terms of attainability sets and their support functions.

1 Introduction

Optimal control of systems with fractional dynamics is a hard problem due to specific of fractional differentiation operators, e.g. lack of the semigroup property. The papers on this topic include [1], where necessary optimality conditions of Euler–Lagrange were derived, and [2], where the problem of time-optimal control is addressed.

Here the fractional time-optimal control problem [2] for a linear system with fractional dynamics is treated using technique of attainability sets and their support functions. This approach has its roots in some methods of the differential games theory [3, 4].

2 Preliminary Results

Denote by \mathbb{R}^n the n -dimensional Euclidean space and by $\mathbb{R}_+ = [0, \infty)$ the positive semi-axis. In what follows we will also denote by $x \cdot y$ the scalar (dot) product and by $\|x\|$ the Euclidean norm for any $x, y \in \mathbb{R}^n$.

Suppose $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is an absolutely continuous function. Let us recall that the Riemann–Liouville (left-sided) fractional integral and derivative of order α , $0 < \alpha < 1$, are defined as follows:

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a,$$

$$D_{a+}^{\alpha} f(t) = \frac{d}{dt} J_{a+}^{1-\alpha} f(t), \quad t > a.$$

In what follows we will omit the lower limit of integration in the notation if it is equal to zero, i.e. $J^\alpha f(t) \triangleq J_{0+}^\alpha f(t)$, $D^\alpha f(t) \triangleq D_{0+}^\alpha f(t)$.

Along with the left-sided fractional integrals and derivatives, one can consider their right-sided counterparts:

$$\begin{aligned} J_{b-}^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau, \quad t < b, \\ D_{b-}^\alpha f(t) &= -\frac{d}{dt} J_{b-}^{1-\alpha} f(t), \quad t < b. \end{aligned}$$

In [5] the Mittag-Leffler generalized matrix function was introduced:

$$E_{\rho,\mu}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho + \mu)}, \quad (1)$$

where $\rho > 0$, $\mu \in \mathbb{C}$, and B is an arbitrary square matrix of order n . It should be noted that $E_{\rho,\mu}(B)$ generalizes the matrix exponential as

$$E_{1,1}(B) = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}. \quad (2)$$

The matrix α -Exponential function, introduced in [6], is closely related to the Mittag-Leffler generalized matrix function:

$$e_\alpha^{At} = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma[(k+1)\alpha]} = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha).$$

The both functions play important role in the theory of fractional differential equations (FDEs). In particular, consider a system of linear FDEs with constant coefficients

$$D^\alpha z = Az + u, \quad 0 < \alpha < 1, \quad (3)$$

where $z \in \mathbb{R}^n$, A is a square matrix, and $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is a measurable and bounded function, under the initial condition

$$J^{1-\alpha} z|_{t=0} = z^0. \quad (4)$$

Then the solution to the Cauchy-type problem (3), (4) can be written down as follows [7]

$$z(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) z^0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^\alpha) u(\tau) d\tau \quad (5)$$

or, in terms of matrix α -exponential function, as [6]

$$z(t) = e_\alpha^{At} z^0 + \int_0^t e_\alpha^{A(t-\tau)} u(\tau) d\tau. \quad (6)$$

Now we proceed with a homogeneous linear system involving right-sided fractional derivative in the sense of Riemann–Liouville

$$D_{b-}^{\alpha} z(t) = Az(t), \quad z \in \mathbb{R}^n, \quad t < b, \quad 0 < \alpha < 1, \quad (7)$$

under the boundary condition

$$J_{b-}^{1-\alpha} z|_{t=b} = \hat{z}. \quad (8)$$

The following lemma holds true.

Lemma 1. *Equation (7) under the condition (8) has a solution given by the following formula*

$$z(t) = \hat{z} e_{\alpha}^{A(b-t)}. \quad (9)$$

Proof. Since $e_{\alpha}^{A(b-t)}$ is an entire function, the corresponding power series can be integrated and differentiated term-by-term. In view of the formulas [8]

$$D_{b-}^{\alpha} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}, \quad (10)$$

$$J_{b-}^{\alpha} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1}, \quad (11)$$

one can easily verify that (9) satisfies (7) by direct substitution. Moreover, (11) implies that $J_{b-}^{1-\alpha} \hat{z} e_{\alpha}^{A(b-t)} = \hat{z} E_{\alpha,1}(A(b-t)^{\alpha})$ and condition (8) is also fulfilled. \square

The following properties of the matrix α -exponential function are direct consequences of properties of the conjugate transpose:

$$(e_{\alpha}^{At})^* = e_{\alpha}^{A^*t} \quad (12)$$

$$\psi \cdot e_{\alpha}^{At} u = e_{\alpha}^{A^*t} \psi \cdot u \quad (13)$$

Below we present some properties [9] of set-valued maps used in the sequel.

Let us suppose that U is a nonempty compact (closed and bounded) set in \mathbb{R}^n . Hereafter we will denote by $U[0, t]$ the set of all measurable functions defined on $[0, t]$ and taking their values in U .

$$\sup_{u(\cdot) \in U[0, t]} \int_0^t f(\tau, u(\tau)) d\tau = \int_0^t \max_{u \in U} f(\tau, u) d\tau \quad (14)$$

Denote by $\text{co } M$ and $\overline{\text{co}} M$ the convex hull and the closure of the convex hull of a set $M \subset \mathbb{R}^n$, respectively.

For any continuous function $F : [0, t] \times U \rightarrow \mathbb{R}^n$, the set-valued map $F(\tau, U)$ possess the following property:

$$\int_0^t F(\tau, U) d\tau = \left\{ \int_0^t F(\tau, U) d\tau : u(\cdot) \in U[0, t] \right\} = \overline{\text{co}} \int_0^t F(\tau, U) d\tau. \quad (15)$$

The integral $\int_0^t F(\tau, U) d\tau$ is to be thought of in the sense of Aumann, i.e. as the set of integrals of all measurable selections of the set valued map $F(\tau, U)$.

Here we recall definition of the support function and present a useful result of convex analysis.

Let $M \in \mathbb{R}^n$ be a convex closed set, i.e. $M = \overline{\text{co}} M$. Then the function

$$\sigma_M(\psi) = \sup_{m \in M} \psi \cdot m, \quad \psi \in \mathbb{R}^n$$

is called the support function of M .

Lemma 2 ([4]). *Let X and M be convex closed sets. Moreover, assume that X is bounded. Then $X \cap M = \emptyset$ if and only if there exist a vector $\psi \in \mathbb{R}^n$ and a number $\varepsilon > 0$ such that*

$$\sigma_X(\psi) + \sigma_M(-\psi) \leq -\varepsilon. \quad (16)$$

Proof. Since the sets X and M are disjoint, by assumptions of the lemma and in view of hyperplane separation theorem, there exist a vector ψ and a number $\varepsilon > 0$ such that

$$\psi \cdot x \leq \psi \cdot m - \varepsilon \quad \forall x \in X, m \in M.$$

Taking supremum in x on the left-hand side and infimum in m on the right-hand side, we obtain an equivalent inequality

$$\sigma_X(\psi) \leq \inf_{m \in M} \psi \cdot m - \varepsilon = - \sup_{m \in M} (-\psi \cdot m) - \varepsilon = -\sigma_M(-\psi) - \varepsilon,$$

which completes the proof. \square

Corollary 1. *Let $X = \overline{\text{co}} X$, $M = \overline{\text{co}} M$ and X be bounded. Then $X \cap M \neq \emptyset$ if and only if*

$$\lambda_{X,M} = \min_{\|\psi\|=1} [\sigma_X(\psi) + \sigma_M(-\psi)] \geq 0. \quad (17)$$

3 Optimal Control Problem

Consider a system of linear fractional differential equations (FDEs) with constant coefficients (3) under the initial condition (4).

Let us fix a point $m \in \mathbb{R}^n$. Here we formulate the optimal control problem: find a control function $u(\cdot)$, $u : \mathbb{R}_+ \rightarrow U$, from a class of measurable functions taking their values in a nonempty compact set U , $U \subset \mathbb{R}^n$, such that the corresponding trajectory of (3), (4) arrives at m in the shortest time T .

If we fix some admissible control function $u(\cdot) \in U[0, t]$, then the solution to the Cauchy-type problem (3), (4) is given by (6).

Consider the attainability set

$$\begin{aligned} Z(t, z^0) &= \left\{ e_\alpha^{At} z^0 + \int_0^t e_\alpha^{A(t-\tau)} u(\tau) d\tau : u(\cdot) \in U[0, t] \right\} \\ &= e_\alpha^{At} z^0 + \int_0^t e_\alpha^{A(t-\tau)} U d\tau. \end{aligned} \quad (18)$$

According to properties of integrals of set-valued maps, in view of (15), the attainability set $Z(t, z^0)$ is closed and convex, while the boundedness of U implies $Z(t, z^0)$ is also bounded.

Consider support function of the attainability set (18).

$$\begin{aligned}\sigma_{Z(t, z^0)}(\psi) &= \sup_{z \in Z(t, z^0)} (z \cdot \psi) \\ &= \sup_{u(\cdot) \in U[0, t]} \left\{ \psi \cdot e_\alpha^{At} z^0 + \int_0^t \psi \cdot e_\alpha^{A(t-\tau)} u(\tau) d\tau \right\} \\ &= \psi \cdot e_\alpha^{At} z^0 + \int_0^t \sigma_U(e_\alpha^{A^*(t-\tau)} \psi) d\tau.\end{aligned}\tag{19}$$

Here we applied properties (12)–(14).

Let us introduce the function

$$\lambda(t, z^0) = \min_{\|\psi\|=1} [\sigma_{Z(t, z^0)}(\psi) - m \cdot \psi]\tag{20}$$

and denote

$$T(z^0) = \min\{t \geq 0 : \lambda(t, z^0) \geq 0\}.\tag{21}$$

Then the following theorem holds true.

Theorem 1. *Trajectory of the system (3), (4) can be brought to the point m at the minimal time $T = T(z^0)$, given by the formula (21), with the help of control function of the form*

$$\hat{u}(\tau) = \arg \max_{u \in U} u \cdot \psi(\tau),$$

where $\psi(\tau)$ is a solution to the adjoint (co-state) system

$$D_{T-}^\alpha \psi = A^* \psi,\tag{22}$$

$$J_{T-}^{1-\alpha} \psi|_{t=T} = \hat{\psi}\tag{23}$$

and $\hat{\psi} = \arg \min_{\|\psi\|=1} [\sigma_{Z(T, z^0)}(\psi) - \psi \cdot m]$.

Proof. Let $T = \min\{t \geq 0 : m \in Z(t, z^0)\}$. Here minimum is attained due to the closedness of $Z(t, z^0)$.

Moreover, m is a boundary point of $Z(T, z^0)$, i.e. $m \in \partial Z(T, z^0)$. As a boundary point, m is contained in at least a supporting hyperplane $H(\hat{\psi}) = \{x \in \mathbb{R}^n : \hat{\psi} \cdot x = \sigma_{Z(T, z^0)}(\hat{\psi})\}$. Hence, for some $\hat{\psi}$

$$\hat{\psi} \cdot m = \sigma_{Z(T, z^0)}(\hat{\psi}).\tag{24}$$

Thus, the control function $\hat{u}(\cdot)$ that ensures bringing trajectory of (3), (4) to the point m is the function at which the maximum in (19) is attained. Therefore it must satisfy

$$\hat{u}(\tau) = \arg \max_{u \in U} u \cdot e_\alpha^{A^*(T-\tau)} \hat{\psi}, \quad \tau \in [0, T].$$

In view of Lemma 1, $\psi(\tau) = e_\alpha^{A^*(T-\tau)}\hat{\psi}$ is a solution to (22).

According to Corollary 1, $m \in Z(t, z^0)$ if and only if $\lambda(t, z^0) \geq 0$, hence $T = T(z^0) = \min\{t \geq 0 : \lambda(t, z^0) \geq 0\}$. Since $\lambda(T, z^0) \geq 0$, in virtue of (24), $\hat{\psi}$ delivers minimum to the expression $\sigma_{Z(T, z^0)}(\psi) - \psi \cdot m$.

Thus

$$\hat{u}(\tau) = \arg \max_{u \in U} u \cdot \psi(\tau),$$

where $\psi(\tau)$ is a solution to (22) under the condition (23), which completes the proof. \square

4 Example

Let us illustrate the above results by a simple example.

Consider a system with fractional dynamics described by the equation

$$D^\alpha z = u, \quad z \in \mathbb{R}^n, \quad \|u\| \leq 1, \quad 0 < \alpha < 1, \quad (25)$$

under the initial condition (4). In this example, the matrix A and U is the unit ball centered at the origin.

Hence

$$e_\alpha^{At} = e_\alpha^{A^*t} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} I$$

and the support function of the attainability set has the form

$$\begin{aligned} \sigma_{Z(t, z^0)}(\psi) &= \psi \cdot e_\alpha^{At} z^0 + \int_0^t \sigma_U(e_\alpha^{A^*(t-\tau)} \psi) d\tau \\ &= \psi \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^0 + \int_0^t \left\| \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \psi \right\| d\tau \\ &= \psi \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} z^0 + \|\psi\| \frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Suppose that $m = 0$, then

$$\lambda(t, z^0) = \min_{\|\psi\|=1} [\sigma_{Z(t, z^0)}(\psi)] = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \|z^0\|. \quad (26)$$

Thus the minimum time T , at which trajectory of the system (25) can reach the origin can be found as the smallest positive root of the equation

$$\frac{t^\alpha}{\Gamma(\alpha+1)} = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \|z^0\|.$$

At $t = 0$ the left-hand side of the latter equation is zero while its right-hand side is infinite, provided that $\|z^0\| \neq 0$. As $t \rightarrow \infty$ the left-hand side increases without bound and the right-hand side approaches zero. Thus, the equation has a positive solution.

The system adjoint to (25), (4) has the form

$$D_{T-}^{\alpha} \psi = 0,$$

$$J_{T-}^{1-\alpha} \psi|_{t=T} = -\frac{z^0}{\|z^0\|}$$

and its solution is

$$\psi(t) = -\frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)} \frac{z^0}{\|z^0\|}.$$

Finally the optimal control that ensures bringing trajectory of (25), (4) to the origin at the minimal time T is

$$u(t) \equiv -\frac{z^0}{\|z^0\|}.$$

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